

Connection between Calogero-Marchioro-Wolfes type few-body models and free oscillators

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Abstract

We establish the exact correspondence of the Calogero-Marchioro-Wolfes model and several of its generalizations with free oscillators. This connection yields the eigenstates and leads to a proof of the quantum integrability. The usefulness of our method for finding new solvable models is then demonstrated by an example.

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Exactly solvable, many-body, interacting systems in one and higher dimensions constitute one of the most exciting branch of active research in physics. Three decades ago, Calogero formulated and solved the quantum mechanics of three identical particles in one-dimension, interacting via pair-wise harmonic and inverse-square potentials [1] and subsequently generalized it to N particles [2]. The connection of this model with random matrices was established by Sutherland who also analysed its thermodynamic behavior [3]. This and related models, popularly known as Calogero-Sutherland (CS) models, have found interesting applications in diverse branches of physics [4]. Models with more than two-body interactions have also attracted considerable attention in the literature. Wolfes has solved the three particle Calogero system in the presence of a three-body interaction of the type $\sum_{i \neq j \neq k \neq i}^3 (x_i + x_j - 2x_k)^{-2}$ [5]; Calogero and Marchioro pointed out the novel aspects of the scattering problem [6]. A number of generalizations of this Calogero-Marchioro-Wolfes (CMW) model have been recently obtained using supersymmetric quantum mechanics [7]. Further, using exchange operator formalism, the algebra of the CMW model has been shown to be D_6 -extended Heisenberg algebra [8]. Note that, for pure two-body and three-body cases, the corresponding symmetries are S_3 and D_3 -extended Heisenberg algebras respectively. The striking features of many of these models are the harmonic oscillator type excited spectra and the coupling dependence of the ground-state energy.

Recently, two of us have shown that the N -particle CS models can be mapped identically to free harmonic oscillators by a similarity transformation (ST) [9,10]. The fact that the energy spectra of both CS and CMW models are identical in their structure motivates us to search for a possible mapping of the CMW type models to free harmonic oscillators. The purpose of this note is to show that such an exact mapping indeed exists. The importance of this result lies in the fact that, it naturally explains why CMW type models have a linear spectra and other properties like quantum integrability. This equivalence leads to the explicit construction of the wavefunctions, starting from the symmetrized eigenfunctions of the free harmonic oscillators. Furthermore, it paves the way for non-trivial generalizations of the CMW model to more than three particles. We demonstrate this fact by explicitly

constructing a new, solvable four particle interacting model which is quantum integrable. Further, we consider several 3-body problems with oscillator-like spectrum and show that all of them can be mapped to a set of decoupled oscillators.

We begin with the Calogero-Marchioro-Wolfes system given by the Hamiltonian ($\hbar = \omega/2 = 2m = 1$)

$$H = - \sum_{i=1}^3 \partial_i^2 + \sum_{i=1}^3 x_i^2 + g \sum_{\substack{i,j=1 \\ i \neq j}}^3 \frac{1}{x_{ij}^2} + 3f \sum_{\substack{i,j=1 \\ i \neq j}}^3 \frac{1}{y_{ij}^2} \quad , \quad (1)$$

here, $\partial_i \equiv \frac{\partial}{\partial x_i}$, $x_{ij} \equiv x_i - x_j$, $y_{ij} \equiv x_i + x_j - 2x_k$; $i \neq j \neq k \neq i$ and $g > -\frac{1}{4} < f$ are the coupling constants.

The ground-state wavefunction is given by

$$\psi_0 = G |X|^\alpha |Y|^\lambda \quad , \quad (2)$$

where, $G \equiv \exp\{-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)\}$, $X \equiv x_{12}x_{23}x_{31}$, $Y \equiv y_{12}y_{23}y_{31}$, $g = \alpha(\alpha - 1)$ and $f = \lambda(\lambda - 1)$. Now, one can perform the following ST:

$$\tilde{H} \equiv \psi_0^{-1} H \psi_0 = 2 \sum_{i=1}^3 x_i \partial_i - \hat{A} + E_0 \quad , \quad (3)$$

where, $E_0 \equiv 3(2\alpha + 2\lambda + 1)$ and $\hat{A} \equiv \sum_{i=1}^3 \partial_i^2 + 2\alpha \sum_{i < j} \frac{1}{x_{ij}} \hat{d}_{ij} + 2\lambda \sum_{i < j} \frac{1}{y_{ij}} \hat{D}_{ij}$; here $\hat{d}_{ij} \equiv \partial_i - \partial_j$ and $\hat{D}_{ij} \equiv \partial_i + \partial_j - 2\partial_k$.

Since the Euler operator $\sum_i x_i \partial_i$ measures the degree of any homogeneous function of x_i and ∂_i , it is easy to verify the commutation relation

$$[2 \sum_i x_i \partial_i \quad , \quad \exp\{-\frac{1}{4}\hat{A}\}] = \hat{A} \exp\{-\frac{1}{4}\hat{A}\} \quad . \quad (4)$$

The above result can be used to make another ST on \tilde{H} , which yields

$$\bar{H} \equiv \exp\{\frac{1}{4}\hat{A}\} \tilde{H} \exp\{-\frac{1}{4}\hat{A}\} = 2 \sum_i x_i \partial_i + E_0 \quad . \quad (5)$$

One more ST on \bar{H} by $\hat{T} \equiv \exp\{\frac{1}{4}\sum_i \partial_i^2\} G^{-1}$ establishes the connection of the CMW Hamiltonian with those of free harmonic oscillators, *i.e.*,

$$\hat{T}^{-1} \bar{H} \hat{T} = \sum_{i=1}^3 (-\partial_i^2 + x_i^2) + (E_0 - 3) \quad . \quad (6)$$

It is worth mentioning that for the normalizability of the wavefunctions, one needs to check that the action of $\exp\{-\hat{A}/4\}$ on an appropriate linear combination of the eigenstates of $\sum_{i=1}^3 x_i \partial_i$ yields a polynomial solution. We observe that the above example is an explicit realization of the more general result established earlier in [9]: all D dimensional N particle Hamiltonians which can be brought through a suitable transformation to the generalized form, $\tilde{H} = \sum_{l=1}^D \sum_{i=1}^N x_i^{(l)} \frac{\partial}{\partial x_i^{(l)}} + E_0 + \hat{A}$ can also be mapped to $\sum_{l=1}^D \sum_{i=1}^N x_i^{(l)} \frac{\partial}{\partial x_i^{(l)}} + E_0$ by $\exp\{-d^{-1}\hat{A}\}$; where, the operator \hat{A} is any homogeneous function of $\frac{\partial}{\partial x_i^{(l)}}$ and $x_i^{(l)}$ with degree d and E_0 is a constant. It should be emphasized that, this procedure reproduces the linear part of the spectrum of the original Hamiltonian.

For the purpose of constructing the eigenfunctions, one can make use of (5). It is interesting to note that, although the monomials $\prod_{i=1}^3 x_i^{m_i}$ are the eigenfunctions of \bar{H} with eigenvalues $E_{m_1, m_2, m_3} \equiv 2(m_1 + m_2 + m_3) + E_0$, they are not acceptable as the eigenfunctions of the CMW Hamiltonian because, the action of $\exp\{-\frac{1}{4}\hat{A}\}$ on them do not yield polynomial solutions. However, the powers of the symmetric combinations, $R \equiv \frac{1}{3} \sum_{i=1}^3 x_i$, $r^2 \equiv \frac{1}{3} \sum_{i < j} x_{ij}^2 = \frac{1}{9} \sum_{i < j} y_{ij}^2$ and $Y \equiv y_{12}y_{23}y_{31}$ in the form $R^{n_1} (r^2)^{n_2} Y^{2n_3}$, are not only the eigenfunctions of \bar{H} with eigenvalues $E_{n_1, n_2, n_3} \equiv 2(n_1 + 2n_2 + 6n_3) + E_0$ but also yield polynomial solutions upon the action of $\exp\{-\frac{1}{4}\hat{A}\}$; here, $n_1, n_2, n_3 = 0, 1, 2, \dots$. Therefore, the normalizable wavefunctions of the CMW Hamiltonian H , in the Cartesian basis, are given by

$$\psi_{n_1, n_2, n_3} = \psi_0 \exp\{-\frac{1}{4}\hat{A}\} \left(R^{n_1} (r^2)^{n_2} Y^{2n_3} \right) \quad . \quad (7)$$

In the following, we explicitly construct some unnormalized eigenfunctions of (1) using (7).

Case I: $n_2 = n_3 = 0$

This case corresponds to center-of-mass degree of freedom:

$$\psi_{n_1, 0, 0} = \psi_0 \exp\{-\frac{1}{4}\hat{A}\} R^{n_1} = \exp\{-\frac{1}{4} \sum_{N=1}^3 \partial_i^2\} R^{n_1} \quad . \quad (8)$$

This can be cast in the form [11],

$$\psi_{n_1,0,0} = 6^{-n_1} n_1! \psi_0 \sum_{\sum_{i=1}^3 m_i = n_1} \prod_{i=1}^3 \frac{H_{m_i}(x_i)}{m_i!} \quad . \quad (9)$$

Here, $H_{m_i}(x_i)$ are the Hermite polynomials.

Case II: $n_1 = n_3 = 0$

Another orthogonal set characterized by the quantum number n_2 can be written in the form,

$$\psi_{0,n_2,0} = \psi_0 \exp\left\{-\frac{1}{4}\hat{A}\right\} (r^2)^{n_2} \quad . \quad (10)$$

It can easily be checked that $\hat{A}(r^2)^n = 4n(n + 3\alpha + 3\lambda)(r^2)^{n-1}$; this gives $\psi_{0,n_2,0}$ as

$$\begin{aligned} \psi_{0,n_2,0} &= \psi_0 (-1)^{n_2} n_2! \sum_{m=0}^{n_2} \frac{(-1)^m}{m!(n_2 - m)!} \frac{(3\alpha + 3\lambda + n_2)!}{(3\alpha + 3\lambda + m)!} (r^2)^m \quad , \\ &= \psi_0 (-1)^{n_2} n_2! L_{n_2}^{3\alpha+3\lambda}(r^2) \quad . \end{aligned} \quad (11)$$

Here, $L_{n_2}^{3\alpha+3\lambda}(r^2)$ is the Lagurre polynomial.

Case III: $n_1 = n_2 = 0$

Now, (7) becomes

$$\psi_{0,0,n_3} = \psi_0 \exp\left\{-\frac{1}{4}\hat{A}\right\} Y^{2n_3} \quad . \quad (12)$$

We note that, on the odd powers of Y , the action of $\exp\{-\frac{1}{4}\hat{A}\}$ does not yield polynomial solutions and hence the resulting states are not normalizable. As an example,

$$\hat{A}Y = 2\lambda \frac{1}{Y} (y_{12}^4 + y_{23}^4 + y_{31}^4) \quad . \quad (13)$$

It is clear that the action of $\exp\{-\frac{1}{4}\hat{A}\}$ on Y will contain negative powers of Y and can not be terminated as a polynomial. The wavefunctions can be computed by making use of

$$\hat{A}Y^{2n_3} = 4n_3(3[2n_3 - 1] + \lambda)Y^{2(n_3-1)}(y_{12}^4 + y_{23}^4 + y_{31}^4) \quad . \quad (14)$$

As an example, the $n_3 = 1$ state is given by

$$\psi_{0,0,1} = \psi_0 \left(Y^2 - (3 + \lambda) \sum_{i < j}^3 y_{ij}^4 - \frac{9}{2}(2 + 3\alpha + 3\lambda) \sum_{i < j}^3 y_{ij}^2 - \frac{3}{4}(2 + 3\alpha + 3\lambda)^2 \right) \quad . \quad (15)$$

It should be noted that, in the Jacobi coordinates, the above set of wavefunctions involve angle variable and hence is orthogonal to the former two sets.

The underlying algebraic structure of CMW model can also be found easily from (5), by defining the creation and annihilation operators as $a_i^+ = \hat{S}x_i\hat{S}^{-1}$ and $a_i^- = \hat{S}\partial_i\hat{S}^{-1}$: $[a_i^-, a_j^+] = \delta_{ij}$ and the CMW Hamiltonian becomes

$$H = 2 \sum_i H_i = \sum_i \{a_i^-, a_i^+\} + (E_0 - 3) \quad , \quad (16)$$

where, $\hat{S} \equiv \psi_0 \exp\{-\frac{1}{4}\hat{A}\}$, $H_i \equiv \frac{1}{2}\{a_i^-, a_i^+\} + \frac{1}{6}(E_0 - 3)$ and $[H_i, a_i^-(a_i^+)] = -a_i^-(a_i^+)$. Here, we would like to remark that, the states created by the action of individual a_i^+ on the ground-state $|0\rangle$ which is obtained from $a_i^-|0\rangle = 0$, are not normalizable unlike their free counterparts; however their symmetric combinations (7) are found to be normalizable. This shows that there are no single particle excitations and any excited state will contain all the three particles in some symmetric state. In other words, the present analysis gives an algebraic statement about a truly correlated system.

Now, the integrability of the CMW model can be seen easily. It is obvious that $[H, H_k] = [H_i, H_j] = 0; i, j, k = 1, 2, 3$. Therefore, the set $\{H_1, H_2, H_3\}$ provides the three conserved quantities. From this set, one can construct, three linearly independent symmetric conserved quantities. This proof of integrability is entirely different from the one given in [8]. It is of considerable interest to note that, all the above analyses done for the CMW model will go through even in the absence of pure two-body terms ($\alpha = 0$ or 1), i.e., a model with pure three-body inverse-square interaction also shares the same algebraic structure as that of the original CMW model. It is worth mentioning that, in the limit $g \rightarrow 0$ or $f \rightarrow 0$, CMW model reproduces only a part of the spectrum of the Hamiltonians with pure two-body or three-body inverse-square potentials respectively. This happens due to the presence of the singular interactions [1].

The above technique can be extended to a one parameter family of potentials connected to the CMW model [12]. The Hamiltonian is given by

$$H = -\sum_{i=1}^3 \partial_i^2 + \sum_{i=1}^3 x_i^2 + g \sum_{i \neq j}^3 \frac{1}{(x_{ij} \cos \delta + \frac{1}{\sqrt{3}} y_{ij} \sin \delta)^2} \quad , \quad (17)$$

where, $0 \leq \delta \leq \pi/6$. Performing a ST on the above Hamiltonian by the ground-state wavefunction ψ_0 , one gets

$$H' \equiv \psi_0^{-1} H \psi_0 = 2 \sum_i^3 x_i \partial_i - \hat{B} + E_0 \quad , \quad (18)$$

where, $E_0 = 3 + 6\alpha$, G and α as before, $\psi_0 = GX^\alpha$, $X \equiv \prod_{i < j}^3 X_{ij}$, $X_{ij} \equiv [(x_i - x_j) \cos \delta + \frac{(x_i + x_j - 2x_k)}{\sqrt{3}} \sin \delta]$ and $\hat{B} \equiv \sum_{i=1}^3 \partial_i^2 + 2\alpha \sum_{i < j}^3 \frac{1}{X_{ij}} \hat{D}_{ij}$; here, $\hat{D}_{ij} \equiv (\cos \delta + \frac{1}{\sqrt{3}} \sin \delta) \partial_i + (-\cos \delta + \frac{1}{\sqrt{3}} \sin \delta) \partial_j - (\frac{2}{\sqrt{3}} \sin \delta) \partial_k$ and $i \neq j \neq k \neq i$. It should be noted that ψ_0 interpolates smoothly between the pure two-body and three-body cases for $\delta = 0$ or $\delta = \pi/6$ respectively.

Analogous to (3), (18) can also be mapped to free oscillators, *i.e.*,

$$\tilde{H} \equiv \exp\{\frac{1}{4}\hat{B}\} H' \exp\{-\frac{1}{4}\hat{B}\} = 2 \sum_i^3 x_i \partial_i + E_0 \quad , \quad (19)$$

and

$$\hat{T}^{-1} \tilde{H} \hat{T} = -\sum_{i=1}^3 \partial_i^2 + \sum_{i=1}^3 x_i^2 + (E_0 - 3) \quad , \quad (20)$$

where, \hat{T} is as given earlier. By inverse ST, one can recast the above Hamiltonian in the form of decoupled oscillators as given in (16). Construction of the eigenfunctions and proving the quantum integrability of this model can be carried out in parallel to the CMW model. Similar analyses can also be extended to more generalized potentials of the above form [12].

We list below other non-trivial, interacting, three-body potentials with linear spectra [7],

$$V(x_1, x_2, x_3) = g \sum_{i < j} x_{ij}^{-2} - \frac{f_3}{\sqrt{6}r} \sum_{i < j} \frac{y_{ij}}{x_{ij}^2} \quad ,$$

and

$$V(x_1, x_2, x_3) = 3f \sum_{i < j} y_{ij}^{-2} + \frac{f_3}{3\sqrt{2}r} \sum_{i < j} \frac{x_{ij}}{y_{ij}^2} \quad ,$$

where, r , x_{ij} and y_{ij} are as given earlier. One should note that these potentials contain the variable r explicitly and hence the corresponding quantum mechanical problems can be

better tackled in the polar coordinates. We have checked that these models can also be made equivalent to free oscillators.

Finally, we present an example involving four particles, to demonstrate the usefulness of our method for finding new solvable models. Since, these models can be mapped to a set of free oscillators, their eigenspectra are guaranteed to be linear. The Hamiltonian reads,

$$H = - \sum_{i=1}^4 \partial_i^2 + \sum_{i=1}^4 x_i^2 + \alpha(\alpha-1) \sum_{\substack{i,j=1 \\ i \neq j}}^4 \frac{1}{x_{ij}^2} + 2\lambda(\lambda-1) \sum_{\substack{i,j,k=1 \\ i \neq j \neq k \neq i}}^4 \frac{1}{y_{ijk}^2} \\ + \frac{2}{3} \lambda(\lambda+4\alpha) \frac{1}{Y} \sum_{\substack{i,j,k=1 \\ i \neq j \neq k \neq i}}^4 y_{ijk}^2, \quad (21)$$

here, x_{ij} is as before and $Y \equiv y_{123}y_{234}y_{341}y_{412}$ with, $y_{ijk} \equiv x_i + x_j + x_k - 3x_l$. Here, we would like to remark that, the four-particle system, with pure inverse-square two-body interactions, is integrable, whereas the one with pure three-body interaction is not.

The ground-state wavefunction is given by

$$\psi_0 = \exp\left\{-\frac{1}{2} \sum_{i=1}^4 x_i^2\right\} \prod_{i < j} |x_i - x_j|^\alpha |Y|^\lambda. \quad (22)$$

The equivalence of the above Hamiltonian to a set of free oscillators follows from

$$(\psi_0 \exp\{-\frac{1}{4}\hat{C}\} \hat{T})^{-1} H (\psi_0 \exp\{-\frac{1}{4}\hat{C}\} \hat{T}) = - \sum_{i=1}^4 \partial_i^2 + \sum_{i=1}^4 x_i^2 + (E_0 - 4) \quad , \quad (23)$$

where, $\hat{C} \equiv \sum_{i=1}^4 \partial_i^2 + 2\alpha \sum_{i < j}^4 \frac{1}{x_{ij}} \hat{d}_{ij} + 2\lambda \sum_{i < j < k}^4 \frac{1}{y_{ijk}} \hat{F}_{ijk}$; $\hat{F}_{ijk} \equiv \partial_i + \partial_j + \partial_k - 3\partial_l$ and $E_0 = 4 + 8\alpha + 8\lambda$. \hat{T} , x_{ij} and \hat{d}_{ij} are similar to those given earlier. In parallel to the CMW model, one can construct eigenfunctions, show the harmonic oscillator algebra and prove the quantum integrability for this model. As is clear, this method can be extended to N particle systems. Generalization of this model analogous to the one given in (17) can also be dealt in the same manner.

In conclusion, we have shown that a number of models having complicated few-body interactions but with linear energy spectra like harmonic oscillators, can indeed be made equivalent to free oscillators by similarity transformations. There are no single particle excitations in all these models; the wavefunctions contain all the particles in some symmetric

combination. Although the underlying algebraic structure of all these models is that of free harmonic oscillators, unlike the oscillator case, the individual states generated by the creation operators are not normalizable. Only their symmetric combinations are normalizable. Our analysis gave an algebraic statement about a truly correlated system. We conjecture that, all the correlated physical systems in nature will have these features of the CMW model; however, the underlying algebraic structure may be different from that of decoupled harmonic oscillators.

Our method allows one to prove the quantum integrability in a straightforward manner and to construct new interacting solvable models. It is amusing to note that the ground-states of these models are similar to the ones that describe edge excitations in quantum Hall effect [13]. Since the planar wavefunctions have exact correspondence with their one dimensional counterparts [14], our technique can be of potential use for these physical systems. Extension to other many-body Hamiltonians will also throw light on the structure of these complicated interacting systems [15,16]. Finally, we conjecture that, any N -body problem having a linear eigenvalue spectrum can be reduced to a set of decoupled harmonic oscillators by some suitable transformation.

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